Pitch-class sets and microtonalism

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Introduction

This text addresses some aspects of pitch-class sets, in an attempt to apply the concept to equal-tempered scales other than 12-EDO.

Several questions arise, since there are limitations of application of certain pitch-class sets properties and relations in other scales.

This is a draft which sketches some observations and reflections.

Pitch classes

The pitch class concept itself, assuming octave equivalence and alternative spellings to notate the same pitch, is not problematic provided that the interval between near pitches is enough large so that different pitches can be perceived (usually not a problem for a small number of equal divisions of the octave). That is also limited by the context and other possible factors.

The use of another equivalence interval than the octave imposes revere restrictions to the perception and requires specially designed timbres, for example: strong partials 3 and 9 for the Bohlen-Pierce scale (13 equal divisions of 3:1), or stretched/shrinked harmonic spectra for tempered octaves.

This text refers to the equivalence interval as "octave". Sometimes, the number of divisions will be referred as "module".

Interval classes

There are different reduction degrees:

1) fit the interval inside an octave (13rd = 6th);

2) group complementar (congruent) intervals (6th = 3rd);

In large numbers of divisions of the octave, the listener may not perceive near intervals as "classes" but as "intonations" of a same interval category (major 3rd, perfect 4th, etc.).

Intervals may be ambiguous relative to usual categorization: 19-EDO contains, for example, an interval between a M3 and a P4; 17-EDO contains "neutral" thirds and so on.

Small size differences between intervals and strange intervals may lead the listener to be confused whether the pitches belong to a scale or were taken out from the pitch continuous. That can be musically interesting of not, depending on the compositional aim and realization. Anyway, scales with large step sizes assure the discrete feature which is expected for consistency of pitch-class approach.

Interval perception is reduced for very large intervals, and also depends on register and timbre.

There are yet, obviously, differences regarding simultaneous or successive presentation of the pitches yelding intervals.

Pitch- class sets

Pitch sets can be reduced to pitch-class sets by invoking octave equivalence. That overall corresponds to the traditional concepts of chord inversions and voicing.

But a pitch-class set may be presented in both simultaneous or successive forms. This approach works well, as possible to be perceived as the "same" pc set, for sets with small cardinality (in special 3 and 4).

Large sets usually are used as scales or other sorts of collections from where smaller sets are extracted and combined.

TODO: compare the differences in dissonances between trichord and tetrachord classes inside 2 octaves, to know which ones are more sensible or resistend to inversion and voicing.

Transposition

1) Exact transposition: usually easier to recognize as such

2) Modal transposition: similarity relation

3) Pitch-class set transposition: may be perceived as such depending on the familiarity with the set class (limited to the set class identification ability).

Inversion

For trichords, works as a similarity relation, since the contained 3 intervals are the same for the two sets.

Allen Forte classified inversion-related sets as belonging to the same set class. That makes major and minor triads indistinguishable in his sets list. Larry Solomon divided those classes renaming them as, e.g. 3-11 (minor triad) and 3-11B (major triad).

Similarity relations between inversion-related sets worths further investigation. Forte has found that for any 5-elements and 7-elements pc sets in 12-EDO, either the set is equivalent to its own inversion (e.g. pentatonic scale), either the set shares all but one elements with an inversion of itself.

Another remark: {C,E,F#,G} and {C,Db,Eb,G} are inversion-related. They are characteristic subsets of diatonic modes Lydian and Phrygian, modes that show contrastant or even opposite "moods". But they have in common the modal association, and a perfect triad (major and minor respectively) as subset.

Inversion-related sets have the same interval content.

Interval content

Interval content is usually represented in an interval vector, that is the account of intances of every interval class in the set.

It may be useful, for unreduced pitch sets, to analize them for interval content with no reduction of intervals in classes (e.g. not equating a M3 with a m6).

The interval vector is useful to know the number of common tones between a set and its transposition.

If the number of divisions of the octave is even, then the interval that divides the octave in the middle causes an special feature: the transposition of this interval by itself yelds the same pitch-class dyad. Thus, the number of common tones is twice the interval vector accounting for that interval.

The interval content can also be considered as a way to know the potential a set has to put certain intervals in relief as subsets. For example, in 12-EDO, {A,Bb,B,C} (vector [321000]) in closed position (or in ascending melodic order) is characterized by semitones; as {Bb,C,A,B} , the whole-tones are prominent (and also the minor 3rd if played in melodic form). However, this set cannot bring intervals of any other classes.

Interval content is specially remarkable concerning to saturation/absence of some interval class. Examples in 12-EDO:

0123 [321000] (chromatic cluster tetrachord - minor 2nds)
048 [000300] (aug5 chord - major 3rds)
027 [010020] (quartal trichord - perfect 4ths)
0369 [004002] (dim7 chord - minor 3rds)

Z-relation

That is Forte's name for the relation between pitch-class sets which are not inversion-related but have the equal interval vectors.

Remarkable Z-mates are, in 12-EDO, 0146 (4-Z15) and 0137 (4-Z29), the all-interval tetrachords. Their interval vector is [111111].

Some complementar hexachords in 12-EDO show Z-relation. That property is important for dodecaphonic music, not because of interval content, but because one cannot build symmetrical series upon Z-hexachords. One could expect similar properties in other even division numbers N, for sets with N/2 elements.

In 22-EDO, there are Z-triples of pentachords (that is: not pairs, but groups of three pc-set classes with equal interval vectors), for example:

Prime form	Interval vector
{0,1,2,10,13}	< 2 1 1 0 0 0 0 1 2 2 1 >
{0,1,9,10,12}	< 2 1 1 0 0 0 0 1 2 2 1 >
{0,2,3,11,12}	< 2 1 1 0 0 0 0 1 2 2 1 >

Inclusion relations

Subsets and supersets.

Interval vector can be generalized to count subsets with more than 2 elements. Thus, trichord content, tetrachord content etc.

Example in 12-EDO: {D,Eb,Ab,A} has interval vector [200022] and trichordal vector [000040000000], considering the 12 set classes defined by Forte (in which inversions belong to the same class). One can see that the 4 trichord subsets of {D,Eb,Ab,A} belong to the same set class. These subsets are 016/056 -- they contain a tritone, a semitone and a perfect fourth/fifth.

The tricordal content of the diatonic scale in 12-EDO is [040423825160]. The only absent trichords (zeroes) are 012 (a chromatic cluster), 014/034 (a major 3rd divided in two unequal parts) and 048 (aug5 chord). It is interesting that these trichords do not contain any tritones or perfect fourths/fifths.

Invariance

TODO

Symmetry

1) Inversional symmetry: a pc set can be mapped to itself by inversion.

2) Transpositional symmetry: a pc set can map itself by transposition. It is impossible for prime number of divisions of the octave.

Highly composite numbers of divisions should present richer possibilities for transpositional symmetry. Examples: 12-EDO, 16-EDO, 18-EDO, 24-EDO.

For instance, 12 can be divided by 2, 3, 4 and 6. Transpositional symmetry examples in 12-EDO are the aug5 (12/3) and dim7 (12/4) chords in 12-EDO, and Messiaen's "limited transposition modes".

Complement

Complement is a ternary relation. Books about pitch-class sets are usually dedicated to music which uses the complete 12-EDO scale, thus the complement related to the aggregate (all 12 tones) was considered convenient. But this is problematic, since it misses important musical applications of the complement concept.

Forte emphasized similarities between complementar sets which are complementar with respect to the aggregate: their interval vectors have a similar profile or are equal for hexachords. Any 5-element sets is embedded inside its complement in transposed or inverted form, with the only exception of 5-Z12 (01356). For example, in 12-EDO, the literal complement of a diatonic scale is a pentatonic scale, and 3 pentatonic scales are contained inside a diatonic scale.

However, one cannot perceive any similarity between a 4-element set and its "complement" in 72-EDO (all the other 68 pitch classes together). Complement with respect to the complete aggregate is only relevant for scales with small number of tones (e.g. 7-EDO, 8-EDO), and only when all the tones are in use.

Therefore, the ternary definition of complement is required, since one should not assume that all pitch-classes are used in a music section or piece.

For instance, in 12-EDO, {A,C,E,G} and {Eb,Gb,Bb,Db} are complementar with respect to the octatone scale, and that may be a significant musical relation when this scale is used. Furtermore, a composition could use the octatonic scale for a given section, and have another section based on the scale {C,D,E,F,G,G#,A,B}, which shares {A,C,E,G} as a common subset, but contains {D,F,G#,B}, the literal complement of that octatonic scale, with respect to the 12-EDO aggregate.

TODO: It is likely that some assymetrical sets cannot be mapped by transposition into subsets of their complements, but only as inversions. That is a thing to analyse.

Similarity relations

TODO (Likely tuning-specific; otherwise, they might be common strutural features)

Modal transposition

That relation is usually dependant to the actual pitch disposition. It is not well represented in the pitch-class set model, although it might be formalizes as a kind of similarity relation -- but it seems that was not formalized yet.

Multiplication

It is originated of stretching/shrinking a pitch set, by multiplying their intervals by a factor. For example, {C,D,E} can be shrinked to {C,C#,D} or {C,Dd,Db} in 24-EDO.

Usually efficient in melodic imitations using small factors.

For pitch-class sets, the number of elements is maintained if the factor is not a divisor of the module (except 1, which lets the set unchanged). For example, in 12-EDO, the special factors are 5 and 7. They map a semitone onto a 4th or 5th. Other factors result in smaller pitch-class sets (2 only can generate up to two pitchs in interval of tritone, 3 only can generate up to 3 pitches as a aug5 triad, etc.)

When the number of divisions N is a prime number, the multiplication of a pc set by any factor (1..N-1) yelds a pc set with the same cardinality, and the operation is reversible.

There may be different factors and sets that result in total invariance. Example: in 19-EDO, {0,1,8}, multiplied by 7, is equal to its transposition by 18 (or -1); multiplied by 8, it yields an inverse form.

In 12-EDO, pc set classes related by multiplication by 5 or 7 have interval vectors which only differ in the number of semitones (ic1) and fourths (ic5). The numbers of ic1 and ic5 are permutated. If the accounts for these interval classes are equal, then the set classes are *Z*-related. Indeed, 0137 (4-Z29) is 0146 (4-Z15) multiplied by 5 and then transposed by 7. There are also pc sets which are invariant under multiplication by 5 or 7, for example: 0127 (interval vector [210021]).

Derivation of pc sets by multiplication may be a way to find sets which are different but have similar structural properties.

Multiplication by -1 is the same as inversion.

Boulez' multiplication

Boulez' multiplication yields large sets from small sets. One set is transposed to the level of each element of the other set.

Boulez has used that derivation proccess in several of his works, like *Le marteau sains mâitre*and *Éclat/Multiples* -- but he usually treats the pitch groups in their respective octave/relative registers, without reducing them to pitch class sets.

Examples:

In 12-EDO, the multiplication of {0,3,6,9} by {0,4,8} is:

0	3	6	9
4	7	10	1
8	11	2	5

The rows are transpositions of {0,3,6,9} and columns are transpositions of {0,4,8}. The union of the sets above is the complete scale (aggregate).

In 17-EDO, the multiplication of {0,5,10,11} by itself is:

0	5	10	11
5	10	15	16
10	15	3	4
11	16	4	5

The union of the sets above is {0,3,4,5,10,11,15,16}.

When a set is multiplied by its inversion, (at least) one pitch is hold invariant, changing its "function" in every transposition.

In the following example, {0,5,10,11} is multiplied by {0,-5,-10,-11}, in 17-EDO. Note that pitch class 0 is present in all rows and columns:

0	5	10	11
12	0	5	6
7	12	0	1
6	11	16	0

Tone rows (serieses)

TODO